SOLUTIONS

MATHEMATICS SPECIALIST
UNIT 3
Section One:
Calculator-free

Semester 1, 2016

Student Number: In Figures

Student Number: In Words _____

_____

Time allowed for this section
Reading time before commencing work: five minutes
Working time for this section: fifty minutes

Materials required/recommended for this section
To be provided by the supervisor
This Question/Answer Booklet
Formula Sheet

To be provided by the candidate
Standard items: pens, pencils, pencil sharpener, eraser, correction fluid/tape, ruler, highlighters

Special items: nil

Important note to candidates

No other items may be used in this section of the examination. It is **your** responsibility to ensure that you do not have any unauthorised notes or other items of a non-personal nature in the examination room. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of exam
Section One: Calculator-free	7	7	50	53	35
Section Two: Calculator-assumed	13	13	100	97	65
			Total	150	100

Instructions to candidates

1. The rules for the conduct of Western Australian external examinations are detailed in the *Year 12 Information Handbook 2016*. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer Booklet.

3. You must be careful to confine your response to the specific question asked and to follow any instructions that are specified to a particular question.

4. Spare pages are included at the end of this booklet. They can be used for planning your responses and/or as additional space if required to continue an answer.
   - Planning: If you use the spare pages for planning, indicate this clearly at the top of the page.
   - Continuing an answer: If you need to use the space to continue an answer, indicate in the original answer space where the answer is continued, i.e. give the page number. Fill in the number of the question that you are continuing to answer at the top of the page.

5. **Show all your working clearly**. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

6. It is recommended that you **do not use pencil**, except in diagrams.

7. The Formula Sheet is **not** to be handed in with your Question/Answer Booklet.

Section One: Calculator-free **35% (53 Marks)**

This section has **seven (7)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time for this section is 50 minutes.

Question 1 **(5 marks)**

Use the fact that the polynomial $P(x) = x^5 + x^3 - 2x$ is divisible by $D(x) = x^2 - 1$, to find all real and complex numbers that satisfy the equation:

$$x^5 - x = x - x^3$$

(5 marks)

Solution
Since $P(x)$ is monic it can be factorised as $P(x) = x(x^2 - 1)(x^2 + bx + c)$ ✓ Then, by expanding and comparing coefficients: $\therefore P(x) = x^5 + bx^4 + (c-1)x^3 - bx^2 - cx = x^5 + x^3 - 2x$ $\therefore b = 0$ and $c = 2$ ✓✓ $\therefore P(x) = x(x^2 - 1)(x^2 + 2)$ ✓ (Long division of polynomials is also an alternative) Also, $x^5 - x = x - x^3$ becomes $x^5 + x^3 - 2x = 0 = P(x)$ and therefore the solutions are: $x = 0, \pm 1$ and $\pm\sqrt{2}\,i$ ✓
Specific behaviours
✓ Write $P(x)$ in the factorised form as $P(x) = x(x^2 - 1)(x^2 + bx + c)$ ✓ ✓ Determine values of a and b. ✓ Write $P(x)$ in the factorised form as $P(x) = x(x^2 - 1)(x^2 + 2)$ ✓ Clearly acknowledges all five solutions

See next page

Question 2 **(6 marks)**

A sphere has equation $x^2 + y^2 + z^2 - 2x + 4y + 3z + 1 = 0$.

(a) Determine the coordinates of the centre and the radius of the sphere. (3 marks)

Solution
$(x-1)^2 + (y+2)^2 + (z+1.5)^2 = -1 + 1 + 4 + 1.5^2$ $= \frac{16+9}{4} = \left(\frac{5}{2}\right)^2$ Radius is 2.5 units Centre at (1, -2, -1.5)
Specific behaviours
✓ factorises left hand side ✓ states the radius ✓ states centre

(b) Determine the vector equation of the straight line that passes through the points on the sphere where $y = -2$ and $z = 0$. (3 marks)

Solution
$x^2 + 4 - 2x - 8 + 1 = 0$ $x^2 - 2x - 3 = 0$ $(x+1)(x-3) = 0 \Rightarrow x = -1,\ 3$ Point on line is (3, -2, 0) Direction of line is $\langle 1, 0, 0 \rangle$ $\mathbf{r} = 3\mathbf{i} - 2\mathbf{j} + \lambda\mathbf{i} = (3+\lambda)\mathbf{i} - 2\mathbf{j}$
Specific behaviours
✓ determines x-coordinates of points on sphere ✓ states direction of line ✓ states vector equation of line

See next page

Question 3 **(7 marks)**

(a) Let $z = 2\cos\left(\frac{2\pi}{3}\right) + 2i\sin\left(\frac{2\pi}{3}\right)$.

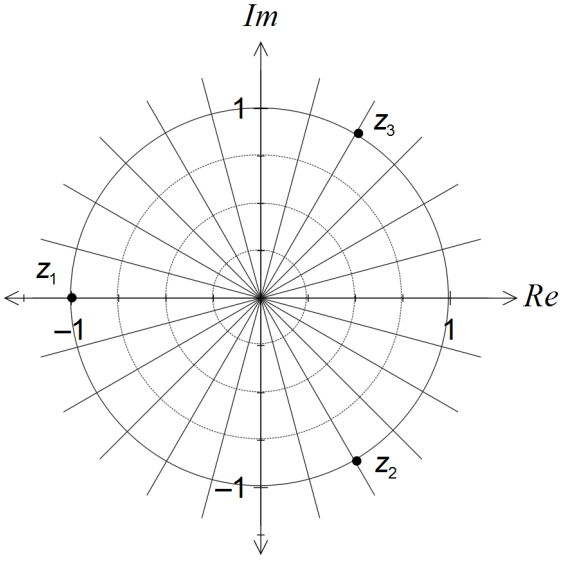
(i) Express z in Cartesian form. (2 marks)

Solution
$z = -1 + \sqrt{3}i$
Specific behaviours
✓ real part ✓ imaginary part

(ii) Determine z^5 in Cartesian form. (3 marks)

Solution
$z^5 = 2^5 cis\left(\frac{2\pi}{3} \times 5\right)$ $= 32cis\left(\frac{10\pi}{3}\right)$ $= 16 \times 2cis\left(-\frac{2\pi}{3}\right)$ $= 16 \times \overline{z}$ $= -16 - 16\sqrt{3}i$
Specific behaviours
✓ uses polar form to determine modulus ✓ uses polar form to determine argument $-\pi < \theta \leq \pi$ ✓ converts to Cartesian form

(b) If $w^3 + 1 = 0$, sketch the location of all roots of this equation on the axes below. (2 marks)



Solution
See diagram - evenly spaced points on circle
Specific behaviours
✓ Show real root at -1 ✓ Shows second and third root around circle

See next page

Question 4 **(15 marks)**

(a) Determine the vector equation of the plane that contains the points A(1, -1, 2), B(2, 1, 0) and C(3, -1, 1). (4 marks)

Solution
$\mathbf{AB} = \langle 1, 2, -2 \rangle$ $\mathbf{AC} = \langle 2, 0, -1 \rangle$ $\mathbf{AC} \times \mathbf{AB} = \langle 2, 3, 4 \rangle$ $\mathbf{r} \quad \langle 2, 3, 4 \rangle = \langle 2, 1, 0 \rangle \ \langle 2, 3, 4 \rangle$ $\mathbf{r} \quad \langle 2, 3, 4 \rangle = 7$
Specific behaviours
✓ finds two vectors in plane ✓ calculates cross product of two vectors ✓ substitutes into vector equation of plane ✓ simplifies vector equation

(b) Plane Π has equation $x + 2y - z = 3$. Line L is perpendicular to Π and passes through the point (1, -6, 4). Determine where line L intersects plane Π. (4 marks)

Solution
$\mathbf{r}_P \quad \langle 1, 2, -1 \rangle = 3$ $\mathbf{r}_L = \langle 1, -6, 4 \rangle + t \langle 1, 2, -1 \rangle$ $\langle 1 + t, 2t - 6, 4 - t \rangle \ \langle 1, 2, -1 \rangle = 3$ $1 + t + 4t - 12 - 4 + t = 3$ $6t = 18 \ \Rightarrow \ t = 3$ $\mathbf{r} = \langle 1, -6, 4 \rangle + 3 \langle 1, 2, -1 \rangle$ $= \langle 4, 0, 1 \rangle \ \Rightarrow$ At $(4, \ 0, \ 1)$
Specific behaviours
✓ writes vector equation of plane ✓ writes vector equation of line through point ✓ substitutes line into plane and solves for t ✓ determines coordinates of point

See next page

(c) The shortest distance between the point Q with position vector $\langle 4,3,k\rangle$ and the plane $\boldsymbol{r}\cdot\langle 1,2,2\rangle = 3$ is 3. Determine the value of k.

(4 marks)

Solution

1. Select any point on the plain say: (1,1,0), Call it point P.
2. Define vector $\overrightarrow{PQ} = (4,3,k) - (1,1,0) = \langle 3,2,k\rangle$
3. Use a formula for the distance of a point from a plane using vector projection:

$d = \frac{\overrightarrow{PQ}\cdot\tilde{n}}{|\tilde{n}|}$

$$d = \frac{\langle 3,2,k\rangle\cdot\langle 1,2,2\rangle}{|\langle 1,2,2\rangle|}$$

$$3 = \frac{3+4+2k}{3}$$

$$\therefore \quad k = 1$$

Specific behaviours

✓ Select any point on the plane
✓ Calculate vector joining selected point and point Q.
✓ Apply formula for the distance of the point from the plane using vector projection
✓ calculate k

(d) Determine distance between two parallel planes Π_1 and Π_2.
Where Π_1: $r\cdot\langle -1,2,-2\rangle = 9$ and Π_2: $r\cdot\langle 2,-4,4\rangle = 24$.

(3 marks)

Solution

1. Distance of Π_1 from the origin:

$$d_1 = \frac{9}{|\langle -1,2,-2\rangle|} = 3$$

2. Distance of Π_2 from the origin:

$$d_2 = \frac{24}{|\langle 2,-4,4\rangle|} = 4$$

The origin (0,0,0) is between Π_1 and Π_2 Therefore the distance between given planes
d = 7

Specific behaviours

✓ Determine the distance of Π_1 from the origin.
✓ Determine the distance of $\Pi_2$from the origin.
✓ Correctly state the distance between planes.

See next page

Question 5 **(7 marks)**

Consider the following system of equations, where k is a real constant.

$$\begin{aligned} x + 2y + z &= 3 \\ 2x - y - 3z &= k \\ x + 3y + kz &= 6 \end{aligned}$$

(a) Solve the system of equations when $k = 1$. (3 marks)

Solution

$$\begin{aligned} x + 2y + z &= 3 \quad (1) \\ 2x - y - 3z &= 1 \quad (2) \\ x + 3y + z &= 6 \quad (3) \\ y &= 3 \quad (3) - (1) \end{aligned}$$

$$\begin{aligned} x + z &= -3 \\ 2x - 3z &= 4 \\ 5x &= -5 \Rightarrow x = -1,\ z = -2 \end{aligned}$$

$$x = -1,\ y = 3,\ z = -2$$

Specific behaviours

✓ eliminates x and z to find y
✓ eliminates and solves for another variable
✓ states values of all three variables

(b) Show that no value of k exists for the system of equations to represent three planes intersecting in a single straight line. (4 marks)

Solution

$$\begin{matrix} 2(1) - (2) \to (2) \\ (3) - (1) \to (3) \end{matrix} : \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 5 & 5 & 6-k \\ 0 & 1 & k-1 & 3 \end{bmatrix}$$

$$5(3) - (2) \to (3) : \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 5 & 5 & 6-k \\ 0 & 0 & 5k-10 & k+9 \end{bmatrix}$$

For infinite solns require $5k - 10 = 0 \Rightarrow k = 2$
and $k + 9 = 0 \Rightarrow k = -9$. Hence no value of k exists.

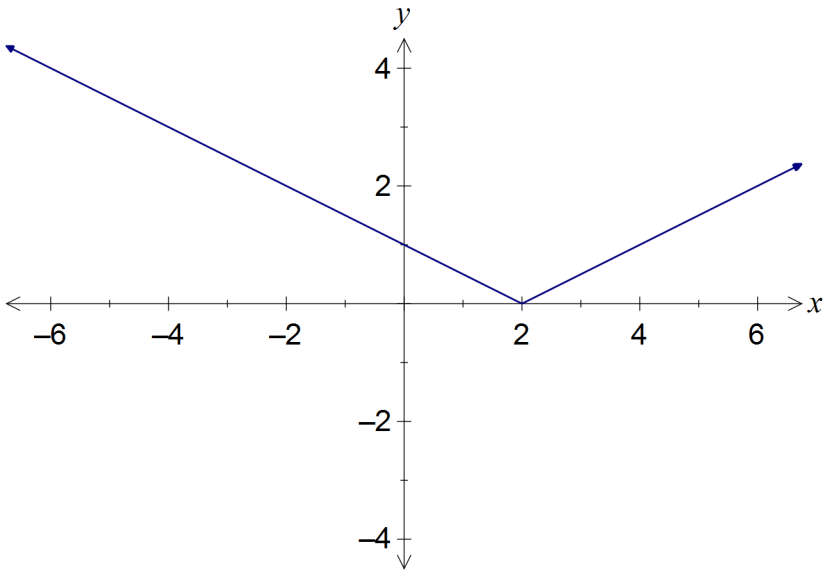
Specific behaviours

✓ reduces second and third rows in initial matrix
✓ reduces third row in second matrix
✓ indicates condition for planes to intersect in single straight line
✓ shows that no value of k exists

See next page

Question 6 **(7 marks)**

(a) Sketch the graph of $y = \dfrac{|x-2|}{2}$ on the axes below. (2 marks)



Solution
See diagram
Specific behaviours
✓ v-shape with cusp at (2, 0)
✓ correct y-intercept

(b) Solve the equation $4|x-8| = 38 - x$. (3 marks)

Solution
$x \geq 8 \Rightarrow 4x - 32 = 38 - x \Rightarrow 5x = 70 \Rightarrow x = 14$ $x < 8 \Rightarrow -4x + 32 = 38 - x \Rightarrow 3x = -6 \Rightarrow x = -2$ $x = -2,\ 14$
Specific behaviours
✓ separates into cases ✓ solves first case ✓ solves second case

(c) Solve the inequality $\dfrac{1}{|x+2|} \leq 1$. (2 marks)

Solution
$\left.\begin{array}{l} x > -2 \Rightarrow 1 \leq x + 2 \Rightarrow x \geq -1 \\ x < -2 \Rightarrow 1 \leq -x - 2 \Rightarrow x \leq -3 \end{array}\right\}$ $x \leq -3,\ x \geq -1$
Specific behaviours
✓ determines correct endpoints ✓ states correct inequalities

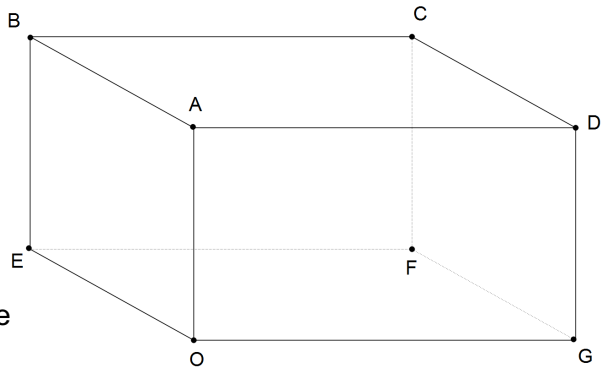
See next page

Question 7 **(6 marks)**

In the rectangular prism shown, $|\overrightarrow{AD}| = 4$ $|\overrightarrow{AB}| = 3$ and $|\overrightarrow{BE}| = 2$.

Let P be the midpoint of BC, and Q be a point on FG such that FQ : QG = 1 : 2.

Let $\overrightarrow{OG}$, $\overrightarrow{OE}$ and $\overrightarrow{OA}$ represent the x, y and z axes respectively, with ***i*, *j*** and ***k*** as their respective unit vectors.



(a) Express each of the following in terms of ***i***, ***j*** and ***k***.

(i) $\overrightarrow{EC}$ (1 mark)

$$\overrightarrow{EC} = \overrightarrow{OC} - \overrightarrow{OE} = \begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix} - \begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ 2 \end{pmatrix} = 4i + 2k$$ ✓

(ii) $\overrightarrow{PG}$ (1 mark)

$$\overrightarrow{PG} = \overrightarrow{OG} - \overrightarrow{OP} = \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \\ -2 \end{pmatrix} = 2i - 3j - 2k$$ ✓

(iii) $\overrightarrow{QA}$ (1 mark)

$$\overrightarrow{QA} = \overrightarrow{OA} - \overrightarrow{OQ} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} - \begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} -4 \\ -2 \\ 2 \end{pmatrix} = -4i - 2j + 2k$$ ✓

(b) Determine the Cartesian equation of the sphere that contains all the vertices of this prism.

(3 marks)

$$\text{Radius} = \frac{1}{2}|\overrightarrow{OC}| = \frac{1}{2}\left|\begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix}\right| = \frac{1}{2}\sqrt{29}$$ ✓

$$\text{Centre at } \frac{1}{2}\overrightarrow{OC} = \frac{1}{2}\begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ \frac{3}{2} \\ 1 \end{pmatrix}$$ ✓

$$\therefore (x-2)^2 + \left(y - \frac{3}{2}\right)^2 + (z-1)^2 = \frac{29}{4}$$ ✓

End of questions

Additional working space

Question number: _____

Additional working space

Question number: _____

Additional working space

Question number: _____

Additional working space

Question number: _____

Additional working space

Question number: _____